A PROOF OF DILWORTH'S DECOMPOSITION THEOREM FOR PARTIALLY ORDERED SETS

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ABSTRACT

A short proof of the following theorem is given: Let P be a finite partially ordered set. If the maximal number of elements in an independent subset of P is k, then P is the union of k chains.

Let P be a partially ordered set. Two elements a and b of P are *comparable* if $a < b$ or $b < a$. A subset C of P is a *chain* if every two distinct elements of C are comparable. A subset S of P is *independent* if no two elements of S are comparable.

The following theorem is due to Dilworth [3, Theorem 1.1]:

THEOREM. *If the maximal number of elements in an independent subset of P is k, then P is the union of k chains.*

This note contains a short proof of Dilworth's theorem for finite sets P.

Proof. Denote by $|P|$ the cardinal of P. The proof proceeds by induction on $|P|$, for all k simultaneously. If $|P| = 1$, there is nothing to prove. Assume, therefore, that the theorem holds for $|P| < n$, and let $|P| = n$. Denote by $P_{\max}$ and $P_{\min}$ the sets of all maximal, resp. minimal elements of P.

CASE 1. P contains an independent subset P_0 of k elements, different from both $P_{\max}$ and $P_{\min}$. Let $P_0 = \{y_1, \dots, y_k\}$ be such a set. Define

$$P^+ = \{x \mid x \in P, (Ey)[y \in P_0 \,\&\, y \leqq x]\},$$

$$P^- = \{x \mid x \in P, (Ey)[y \in P_0 \,\&\, x \leqq y]\}.$$

It is easily verified that $P^+ \cap P^- = P_0$, $P^+ \cap P^- = P$, $P^+ \neq P$ and $P^- \neq P$ (the first relation follows from the independence of P_0, the second from the maximality of P_0, the third ftom $P_0 \neq P_{\min}$ and the fourth from $P_0 \neq P_{\max}$).

Now, $|P^+| < |P|$, $|P^-| < |P|$. By induction hypothesis, P^+ and P^- decompose into k chains:

$$P^+ = \bigcup_{i=1}^{k} U_i, \quad P^- = \bigcup_{i=1}^{k} L_i.$$

The elements of P_0, being the minimal elements of P^+ and the maximal elements

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of P^-, are the minimal elements of the chains U_i and the maximal elements of the chains L_i. Assume, without loss of generality, that y_i is the minimal element of U_i and the maximal element of L_i ($1 \leqq i \leqq k$). Define $C_i = L_i \cup U_i$. C_i is a chain, and we have

$$P = P^- \cup P^+ = \bigcup_{i=1}^{k} C_i.$$

CASE 2. Every independent subset of P containing k elements coincides with $P_{\max}$ or with $P_{\min}$. Take some $a \in P_{\min}$, and choose a $b \in P_{\max}$, such that $b \geqq a$ (b may equal a). Define $C_k = \{a, b\}$, and $P' = P - \{a, b\}$. C_k is a chain, $|P'| < |P|$, and P' contains $k-1$, but no k mutually incomparable elements. Therefore we have, by induction hypothesis, $o' = \bigcup_{i=1}^{k-1} C_i$, where the C_i are chains, and

$$P = P' \cup \{a, b\} = \bigcup_{i=1}^{k} C_i. \qquad \text{Q.E.D.}$$

REMARK. 1. Other proofs of Dilworth's theorem for finite sets may be found in [2], [3], [4] and [5]. The original proof in [3] is direct, but somewhat complicated. The proof in [2] uses the duality theorem of linear programming. In [4], Dilworth's theorem is shown to be equivalent to a theorem of König concerning bi-chromatic graphs ([8, p. 232]). In [5], it is obtained as a consequence of a theorem on the covering of a directed graph by a system of disjoint paths.

REMARK 2. Dilworth's theorem for general sets P can be easily deduced from the finite case, applying the following result, which is a special case of a theorem of Rado ([9], [6], [1]).

THEOREM. *Let P be a set, K a finite set, and let $\mathscr{F}$ be the class of all finite subsets of P. For each $F \in \mathscr{F}$, let ϕ_F be a mapping of F into K. Then there exists a mapping ϕ of P into K, having the following property. For every $F \in \mathscr{F}$ there exists a $G \in \mathscr{F}$, such that $G \supseteq F$ and $\phi(x) = \phi_G(x)$ for all $x \in F$.*

A very short proof of Rado's theorem, using Tychonoff's theorem, may be found in [9]. In [3], the infinite case of Dilworth's theorem is deduced from the finite case by another transfinite argument, using induction on k and Zorn's lemma.

BIBLIOGRAPHY

1. de Bruijn N. G. and Erdös P. 1951, A colour problem for infinite graphs and, a problem in the theory of relations, *Nederl. Akad. Wetensch. Proc. ser. A*, **54,** 371–373.

2. Dantzig G. B. and Hoffman, A. J., 1956, Dilworth's theorem on partially orders sets, Linear inequalities and related systems, *Annals of Mathematics Studies* No. 38, Princeton University Press, pp. 207–214.

3. Dilworth, R. P., 1950, A decomposition theorem for partially ordered sets, *Ann. of Math.* **51,** 161–166.

4. Fulkerson, D. R., 1956, Note on Dilworth's decomposition theorem for partially ordered sets, *Proc. Amer. Math. Soc.*, **7,** 701–702.

5. Gallai, T. and Milgram, A. N., 1960, Verallgemeinerung eines graphentheoretisch-Satzes von Rédei, *Acta Sci. Math.* (*Szeged*), **21**, 181–186.

6. Gottschalk, W. H., 1951, Choice functions and Tychonoff's theorem, *Proc. Amer. Math. Soc.*, **2**, 172.

7. Halmos, P. R., and Vaughan, H. E., 1950, The marriage problem, *Amer. J. Math.*, **72**, 214–215.

8. König, D., 1950, *Theorie der endlichen und unendlichen Graphen*, Chelsea Publishing Co., New York.

9. Rado, R., 1949, Axiomatic treatment of rank in infinite sets, *Canad. J. Math.*, **1**, 337–343.

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